JOURNAL OF APPROXIMATION THEORY 25, 258–265 (1979)

On the Existence of Best Simultaneous Approximation

JAROSLAV MACH

Institut für Angewandte Mathematik der Universität Bonn, 5300 Bonn, West Germany

Communicated by Richard S. Varga

Received October 26, 1977

Let X be a Banach space, V a closed subspace of X, F a bounded subset of X. Let $\mathrm{rad}_V(F) = \inf_{y\in V} \sup_{x\in F} \| x - y \|$, $\mathrm{cent}_V(F) = \{ y \in V; \sup_{x\in F} \| x - y \| = \mathrm{rad}_V(F)\}$. Elements of $\mathrm{cent}_V(F)$ are called best simultaneous approximations to F by elements of V. The problem of the existence, characterization and unicity of such best simultaneous approximations has been recently studied by many authors (see e.g. [11, 12, 13]).

The purpose of this paper is to show that in certain subspaces V of a given Banach space X a best simultaneous approximation exists for every bounded set $F \subset X$. By a simple compactness argument it may be shown that every finite-dimensional subspace of an arbitrary Banach space and every w^*-closed subspace of a dual space have this propeity. In Section 1 we prove that every Weierstrass-Stone subspace of $C(S, X)$, the space of all continuous X-valued functions x on a compact Hausdorff space S equipped with the norm $\| x \| = \sup_{t\in S} | x(t)|$, where $| \cdot |$ is the norm of X, has this property, if X is a uniformly convex Banach space or a space C_σ. This generalises a result of Olech [9] who showed that such subspaces are proximinal, if X is uniformly convex. In Section 2 we show that $\mathrm{cent}_V(F) \neq \varnothing$, if V is an M-ideal in a Lindenstrauss space and F is a compact set. If V is an M-summand in an arbitrary Banach space X admitting Chebyshev centers, then the same is true for every bounded set $F \subset X$. In Section 3, finally, we show that $\mathrm{cent}_V(F) \neq \varnothing$ for every bounded set $F \subset B(S)$, the space of all real-valued bounded functions on a set S and every closed subalgebra V of $B(S)$.

Let $x \in X$, X a Banach space, $r > 0$. We denote by $B(x, r)$ the closed ball of X with center x and radius r. A space C_σ is the space of all real-valued continuous functions x on a compact Hausdorff space S with the property $x(t) = -x(\sigma(t))$ for all $t \in S$, where σ is an involutory homeomorphism of S onto itself. A Lindenstrauss space is a Banach space whose dual is a space $L_1(\mu)$ for some measure μ. All Banach spaces in this paper are real.

0021-9045/79/030258–08$02.00/0

1. Best Simultaneous Approximation by Weierstrass-Stone Subspaces

Let S be a compact Hausdorff space, X a Banach space. A subspace V of $C(S, X)$ is said to be a Weierstrass-Stone subspace, if there is a compact Hausdorff space T and a continuous surjection $: S \to T$ such that $V = \{f \in C(S, X); f = g \circ \text{ for some } g \in C(T, X)\}$. Mazur (unpublished, see e.g. [15, Proposition 7.5.6]), Olech [9] and Blatter [3] have proved that such subspaces are proximinal if $X = \mathbb{R}$, X is uniformly convex, and X is a Lindenstrauss space, respectively. In this section we show that if V is a Weierstrass-Stone subspace of $C(S, X)$ and X is uniformly convex or a space C_σ, then there exists a best simultaneous approximation in V for every bounded set $F \subset C(S, X)$.

Let $B(x, r + \delta)$, $B(y, r + \theta)$, $0 < \theta < \delta$, $r > 0$, $x \in X$, $y \in X$ be two balls in a Banach space X. The following lemma says that, for certain Banach spaces X, it is possible to "move" the center y of the second ball arbitrarily "close" to the center of the first ball x without decreasing the intersection $B(x, r + \theta) \cap B(y, r + \delta)$, if δ is "small" enough. This idea is a modification of that used in Proposition 2 [9].

Lemma 1.1. *Let X be a uniformly convex Banach space or a space C_σ. Let $r > 0$. Then for every $\epsilon > 0$ there is a $\delta(\epsilon)$, $0 < \delta(\epsilon) \leqslant \epsilon$ such that for every $x, y \in X$ there exists an $z_{x,y} \in B(x, \epsilon)$ with the property*

$$B(z_{x,y}, r + \theta) \supset B(x, r + \delta(\epsilon)) \cap B(y, r + \theta)$$

for every θ with $0 < \theta < \delta(\epsilon)$.

Proof. For the case X uniformly convex, a similar argument has been used to prove Lemma 2.1 of [16]. Therefore we omit the proof here.

Let X be a space C_σ. For given $\epsilon > 0$ put $\delta(\epsilon) = \epsilon$. For given $x, y \in X$ it may be easily verified that the function

$$\begin{aligned} z_{x,y}(t) &= x(t) + \epsilon && \text{if} \quad y(t) \geqslant x(t) + \epsilon, \\ &= y(t) && \text{if} \quad |x(t) - y(t)| < \epsilon, \\ &= x(t) - \epsilon && \text{if} \quad y(t) \leqslant x(t) - \epsilon \end{aligned}$$

has the required property.

Remark. It seems not to be easy to decide whether concrete Banach spaces have or have not the property established by Lemma 1.1 for uniformly convex Banach spaces and the spaces C_σ. In particular we do not know whether the spaces $L_1(\mu)$ have this property. Since the proof of our Theorem 1.4 depends only on this property, for a Banach space X an affirmative answer to this question would imply that for every Weierstrass–Stone subspace V of $C(S, X)$ and every bounded set $F \subset (S, X)$ we have $\text{cent}_V(F) \neq \varnothing$.

The next proposition gives an example of a Banach space where the conclusion of Lemma 1.1 does not hold.

PROPOSITION 1.2. *Let $t_k = 1/2k$, $s_k = 1/(2k+1)$, $k \in \mathbb{N}$. Let X be the subspace of $C[0, 1]$ consisting of all functions f which satisfy the relations $f(s_k) = (1/k) f(t_k)$, $k \in \mathbb{N}$. Let $x \equiv 0$. Then for every $n \in \mathbb{N}$, $n \geqslant 3$, there is a y_n such that for every $z \in B(x, 1/10)$ there is a $z_n \in B(x, 1 + 1/n) \cap B(y_n, 1 + 1/2n) \backslash B(z, 1 + 1/2n)$.*

Proof. Given $n \in \mathbb{N}$, $n \geqslant 3$, we define y_n to be 2 in the points t_n and t_{n+1}, 0 in the intervals $[0, t_{n+2}]$, $[s_{n-1}, 1]$, and linear in the intervals $[t_{n+2}, s_{n+1}]$, $[s_{n+1}, t_{n+1}]$, $[t_{n+1}, s_n]$, $[s_n, t_n]$ and $[t_n, s_{n-1}]$. Let $z \in B(x, 1/10)$. Then there is an interval $(u_1, u_2) \subset [t_{n+1}, t_n]$ containing the point s_n such that $z(t) \leqslant 1/5n$ for every $t \in (u_1, u_2)$. Consequently, every $y \in B(z, 1 + 1/2n)$ fulfils $y(t) \leqslant 1 + 7/10n$ for every $t \in (u_1, u_2)$. We find now points $v_1, v_2 \in \mathbb{R}$ such that $(v_1, v_2) \subsetneqq (u_1, u_2)$, $s_n \in (v_1, v_2)$, $y_n(v_i) \leqslant 1 + 1/n$, $i = 1, 2$ (this is possible, since $y_n(s_n) < 1 + 1/n$). Let z_n be $1 + 1/n$ in the intervals $[t_{n+1}, v_1]$, $[v_2, t_n]$, 0 in the intervals $[0, t_{n+2}]$, $[s_{n-1}, 1]$, and linear in $[t_{n+2}, s_{n+1}]$, $[s_{n+1}, t_{n+1}]$, $[v_1, s_n]$, $[s_n, v_2]$, and $[t_n, s_{n-1}]$. It is easy to check that $z_n \in B(x, 1 + 1/n) \cap B(y_n, 1 + 1/2n)$. Since $z_n(t) = 1 + 1/n > 1 + 7/10n$ for every $t \in (u_1, u_2) \backslash (v_1, v_2)$, z_n cannot be in $B(z, 1/10)$. This completes the proof.

Consider a bounded set $F \subset C(S, X)$. For every $t \in T$ denote by $\mathscr{B}(t)$ a neighborhood base of t. For every $U \in \mathscr{B}(t)$ define a set $F_U \subset X$ by

$$F_U = \{x \in X; x = f(s) \text{ for some } f \in F, \text{ some } t' \in U \text{ and some } s \in \varphi^{-1}(t')\}.$$

Put $r_U = \operatorname{rad}_X(F_U)$. For every $t \in T \{r_U\}_{U \in \mathscr{B}(t)}$ is a decreasing net of numbers. Put $r(t) = \lim_U r_U$, $r_F = \sup_{t \in T} r(t)$.

The next lemma gives a lower bound for $\operatorname{rad}_V(F)$.

LEMMA 1.3. *Let X be a Banach space, V a Weierstrass–Stone subspace of $C(S, X)$. Then for every bounded set $F \subset C(S, X)$ we have $\operatorname{rad}_V(F) \geqslant r_F$.*

Proof. Assume the contrary. Then there is a $g \in V$, a $t_0 \in T$ and an $\epsilon_0 > 0$ such that for every $s \in S$ and every $f \in F$ we have

$$\| f(s) - g(s) \| \leqslant r(t_0) - \epsilon_0 .$$

Choose an $s_1 \in \varphi^{-1}(t_0)$. There is a neighborhood U of t_0 such that $\| g(s) - g(s_1) \| \leqslant \epsilon_0/2$ for every $s \in \varphi^{-1}(U)$. Hence

$$\begin{aligned} r(t_0) \leqslant r_U &= \sup_{f \in F} \sup_{t \in U} \sup_{s \in \varphi^{-1}(t)} \| f(s) - g(s_1) \| \\ &\leqslant \sup_{f \in F} \sup_{t \in U} \sup_{s \in \varphi^{-1}(t)} \| f(s) - g(s) \| \\ &\quad + \sup_{t \in U} \sup_{s \in \varphi^{-1}(t)} \| g(s) - g(s_1) \| \leqslant r(t_0) - \epsilon_0/2, \end{aligned}$$

a contradiction.

Now, we prove the main theorem of this section.

THEOREM 1.4. *Let X be a uniformly convex Banach space or a space C_σ. Let V be a Weierstrass–Stone subspace of $C(S, X)$, $F \subset C(S, X)$ a bounded set. Then there exists a $g_0 \in \text{cent}_V(F)$.*

Proof. Define a set-valued mapping $\Phi: T \to 2^X$ by

$$\Phi(t) = \{x \in X; \text{ for any } \epsilon > 0 \text{ there is a } U \in \mathscr{B}(t) \text{ with } F_U \subset B(x, r_F + \epsilon)\}.$$

If h_0 is a continuous selection of Φ, then the function $g_0 = h_0 \circ \varphi$ is obviously the required best simultaneous approximation. We show that the assumptions of Michael's well-known selection theorem are fulfilled.

Let $t \in T$. The set $\Phi(t)$ is obviously closed and convex. We show that $\Phi(t) \neq \varnothing$. Take $\delta(1/2)$ from Lemma 1.1. There exists a neighborhood U_1 of t with $\text{rad}_X(F_{U_1}) < r_F + \delta(1/2)$. Consequently, there is a point $x_1 \in X$ such that $F_{U_1} \subset B(x_1, r_F + \delta(1/2))$. Now, take $\delta(1/4)$ from Lemma 1.1 such that $\delta(1/4) < \delta(1/2)$. There exists a neighborhood $U_2 \subset U_1$ of t and a point $y_1 \in X$ such that $F_{U_2} \subset B(y_1, r_F + \delta(1/4))$. It follows from Lemma 1.1 that there is a point $x_2 \in B(x_1, 1/2)$ such that $F_{U_2} \subset B(x_1, r_F + \delta(1/2)) \cap B(y_1, r_F + \delta(1/4)) \subset B(x_2, r_F + \delta(1/4))$. Continuing this construction inductively, we produce a sequence $\{x_n\} \subset X$, a decreasing sequence of positive numbers $\delta(1/2^n)$ and a sequence $\{U_n\}$ of neighborhoods of t with the properties: $x_{n+1} \in B(x_n, 1/2^n)$, $F_{U_n} \subset B(x_n, r_F + \delta(1/2^n))$, $n \in \mathbb{N}$, and $\lim \delta(1/2^n) = 0$. The sequence $\{x_n\}$ being a Cauchy sequence, we denote its limit by x_0. Obviously $x_0 \in \Phi(t)$.

We complete the proof by showing that Φ is lower semicontinuous. Let $t_0 \in \{t; \Phi(t) \cap G \neq \varnothing\}$ for any open set $G \subset X$. Then there is a point $x \in X$ and a number $\epsilon > 0$ such that $x \in \Phi(t_0)$ and $B(x, \epsilon) \subset G$. Take $\delta(\epsilon)$ from Lemma 1.1. There exists a neighborhood U_1 of t_0 with $F_{U_1} \subset B(x, r_F + \delta(\epsilon))$. We show that $U_1 \subset \{t; \Phi(t) \cap G \neq \varnothing\}$. Let $s \in U_1$, $y \in \Phi(s)$. By Lemma 1.1 there exists a point $z_{x,y} \in B(x, \epsilon) \subset G$ with the property $B(x, r_F + \delta(\epsilon)) \cap B(y, r_F + \theta) \subset B(z_{x,y}, r_F + \theta)$ for every θ with $0 < \theta < \delta(\epsilon)$. It is easy to see that $z_{x,y} \in \Phi(s)$. Indeed, given θ with $0 < \theta < \delta(\epsilon)$, there is a neighborhood U_2 of s such that $U_2 \subset U_1$ and $F_{U_2} \subset B(y, r_F + \theta)$. Since $F_{U_2} \subset F_{U_1}$, we have $F_{U_2} \subset B(x, r_F + \delta(\epsilon)) \cap B(y, r_F + \theta) \subset B(z_{x,y}, r_F + \theta)$.

2. BEST SIMULTANEOUS APPROXIMATION BY M-IDEALS

In this section we establish two theorems on the existence of the best simultaneous approximation in M-ideals. The concept of an M-ideal has been introduced and studied in the important paper [1] of Alfsen and Effros. A closed subspace V of a Banach space X is said to be an M-ideal, if there is a projection P on the dual X^* of X onto $M^\perp$, the annihilator of M, with the property $\| u \| = \| Pu \| + \| u - Pu \|$ for every $u \in X^*$. For further investigations on M-ideals see also [2, 5, 6, 8]. A special case of an M-ideal is an M-summand which is the range of an M-projection P, i.e., a projection with the property $\| x \| = \mathrm{Max}(\| Px \|, \| x - Px \|)$ for every $x \in X$. A Banach space X is said to admit Chebyshev centers if $\mathrm{cent}_X(F) \neq \varnothing$ for every bounded set $F \subset X$.

In the following lemma a property of M-ideals in Lindenstrauss spaces is established.

LEMMA 2.1. *Let X be a Lindenstrauss space, $V \subset X$ an M-ideal, $K \subset X$ a compact set, $r > 0$. Let $B(x, r) \cap V \neq \varnothing$ for every $x \in K$. Let $\bigcap_{x \in K} B(x, r) \neq \varnothing$. Then $\bigcap_{x \in K} B(x, r) \cap V \neq \varnothing$.*

Proof. For every $n \in \mathbb{N}$ let $K_n \subset K$ be a finite $1/n$-net such that $K_n \subset K_{n+1}$. By Theorem 2.17 and Proposition 6.5 of [8] and Theorem 5.8 of [1] there is a $y_1 \in \bigcap_{x \in K_1} B(x, r) \cap V$. Now, assume that for an $n \in \mathbb{N}$ the points $y_i \in \bigcap_{x \in K_i} B(x, r) \cap \bigcap_{j=1}^{i-1} B(y_j, 1/j) \cap V$, (we make use of the convention $\bigcap_{j=1}^{0} B(y_j, 1/j) = X$ here), $i = 1,..., n$, have already been constructed. Then $y_i \in B(x, r + 1/i)$ for $i = 1,..., n$ and every $x \in K$. This implies that the balls $B(y_i, 1/i)$, $i = 1,..., n$, $B(x, r)$, $x \in K$, pairwise intersect. By a well-known theoem of Lindenstrauss (cf., e.g., [7], Sect. 21, Theorem 6) we have $\bigcap_{x \in K_{n+1}} B(x, r) \cap \bigcap_{j=1}^{n} B(y_j, 1/j) \neq \varnothing$. Again by Theorem 2.17 and Proposition 6.5 of [8] and Theorem 5.8 of [1] there is a point $y_{n+1} \in \bigcap_{x \in K_{n+1}} B(x, r) \cap \bigcap_{j=1}^{n} B(y_j, 1/j) \cap V$. Then $y_{n+1} \in B(x, r + 1/(n + 1))$ for every $x \in K$. In this way we construct a sequence $\{y_n\}$ of points of V with the properties: $y_n \in B(x, r + 1/n)$ for every $x \in K$ and every $n \in \mathbb{N}$, $y_m \in B(y_n, 1/n)$ for every $m, n \in \mathbb{N}$ with $m > n$. It is easy to show that $\{y_n\}$ is a Cauchy sequence. Indeed, given $\epsilon > 0$ choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \epsilon/2$. Then, for any $n, m \in \mathbb{N}$ with $n, m > n_0$ we have $\| y_n - y_m \| \leqslant \| y_n - y_{n_0} \| + \| y_{n_0} - y_m \| \leqslant 2/n_0 < \epsilon$. Put $y = \lim y_n$. Then $y \in V \cap \bigcap_{x \in K} B(x, r)$.

THEOREM 2.2. *Let X be a Lindenstrauss space, $V \subset X$ an M-ideal, $F \subset X$ a compact set. Then there exists a $y_0 \in \mathrm{cent}_V(F)$.*

Proof. We obviously have $\mathrm{cent}_X(F) = \bigcap_{n \in \mathbb{N}} \bigcap_{x \in F} B(x, \mathrm{rad}_X(F) + 1/n)$. Since the balls $B(x, \mathrm{rad}_X(F) + 1/n)$, $x \in F$, $n \in \mathbb{N}$, pairwise intersect, it follows from the already mentioned theorem of Lindenstrauss [7, Sect. 21, Theorem 6] that $\mathrm{cent}_X(F) \neq \varnothing$. Put $r = \mathrm{Max}(\mathrm{rad}_X(F), \sup_{x \in F} \mathrm{dist}(x, V))$. Then obviously $\mathrm{rad}_V(F) \geqslant r$. Since the set $\bigcap_{x \in F} B(x, r)$ contains $\mathrm{cent}_X(F)$ it is nonempty. Since V is proximinal [5], we have $B(x, r) \cap V \neq \varnothing$ for every $x \in F$. The assertion follows then from Lemma 2.1.

For M-summands we prove the following.

THEOREM 2.3. *Let X be a Banach space admitting centers, $V \subset X$ an M-summand, $F \subset X$ a bounded set. Then $\mathrm{cent}_V(F) \neq \varnothing$.*

Proof. Let P be an M-projection onto V. We obviously have $\text{rad}_V(F) = \text{Max}(\text{rad}_V(PF), \sup_{x \in F} \| x - Px \|)$. If $\sup_{x\in F} \| x - Px \| > \text{rad}_V(PF)$, every $y_0 \in V$ with $\sup_{x \in F} \| Px - y_0 \| < \sup_{x\in F} \| x - Px \|$ belongs to $\text{cent}_V(F)$. If $\sup_{x\in F} \| x - Px \| \leqslant \text{rad}_V(PF)$, take any $y_0 \in \text{cent}_X(F)$. It is easy to see that $Py_0 \in \text{cent}_V(F)$.

3. Best Simultaneous Approximation by Closed Subalgebras

In [11] Smith and Ward proved that $\text{cent}_V(F) \neq \varnothing$ for every closed subalgebra V of $C(T)$, the space of all continuous real-valued functions on a compact Hausdorff space T and every bounded set $F \subset C(T)$. It follows from their result that the same is true for every closed subalgebra V of $B(S)$ and every bounded set $F \subset B(S)$, where S is an arbitrary set and $B(S)$ the space of all real-valued bounded functions on S equipped with the norm of uniform convergence when considering the space $C(\beta S)$, where βS is the Stone–Čech compactification of S equipped with the discrete topology. The space $C(\beta S)$ is obviously linearly isometric and algebraically isomorphic to $B(S)$.

In this section we give another proof of this fact using a modification of a technique due to Blatter and Seever [4].

Let S be a set, V a subalgebra of $B(S)$. Define a set W by $W = \{x \in B(S);\ x(t) = y(t) + \alpha$ for all $t \in S$, some $y \in V$ and $\alpha \in R\}$. Clearly W is a closed lattice cone, i.e., a convex cone in $B(S)$, containing the constant functions and closed both in the topology of $B(S)$ and under lattice operations. W defines a binary relation δ on 2^S, called quasi-proximity: $A\delta B$ iff there does not exist an $x \in W$ such that $\chi_A \leqslant x \leqslant \chi_{X\setminus B}$, where χ_A and $\chi_{X\setminus B}$ are the characteristic functions of A and $X\setminus B$, respectively. It follows from the Interposition theorem 2.2 and the Characterization theorem 2.3 [4] that if for some $x, y \in B(S)$ and every $r, s \in \mathbb{R}$ with $r < s$ we have $\{t; x(t) \geqslant s\}$ non $\delta\{t; y(t) < r\}$, then there is a $z \in W$ with $x \leqslant z \leqslant y$.

Theorem 3.1. *Let S be an arbitrary set, V a subalgebra of $B(S)$, $F \subset B(S)$ a bounded set. Then* $\text{cent}_V(F) \neq \varnothing$.

Proof. Let $R = \{t \in S; x(t) = 0$ for all $x \in V\}$. Clearly $V = \{x \in W;\ x(t) = 0$ for all $t \in R\}$. We first prove that $\text{rad}_V(F) \geqslant c$, where $c = \text{Max}(a_1, a_2, a_3)$ with

$$a_1 = \sup_{A\delta B} (1/2)(\inf_{t\in A} \sup_{x\in F} x(t) - \sup_{t\in B} \inf_{x\in F} x(t)),$$

$$a_2 = \sup_{A\delta R} \inf_{t\in A} \sup_{x\in F} x(t),$$

$$a_3 = \sup_{R\delta B} (-\sup_{t\in B} \inf_{x\in F} x(t)).$$

For fixed $y \in V$ denote $d(y) = \sup_{x \in F} \| x - y \|$. We have $x(t) - y(t) \leqslant d(y)$ for all $x \in F$ and all $t \in S$ which implies

$$\sup_{x \in F} x(t) - y(t) \leqslant d(y), \qquad t \in S. \tag{3.2}$$

Similarly we have

$$y(t) - \inf_{x \in F} x(t) \leqslant d(y), \qquad t \in S. \tag{3.3}$$

Let $A \delta B$. Then, by Proposition 4.1 of [4], (3.2) and (3.3), we have

$$\inf_{t \in A} \sup_{x \in F} x(t) - d(y) \leqslant \inf_{t \in A} y(t) \leqslant \sup_{t \in B} y(t) \leqslant \sup_{t \in B} \inf_{x \in F} x(t) + d(y).$$

It follows $a_1 \leqslant d(y)$.

Let $A \delta R$. Proposition 4.1 of [4] and (3.2) imply

$$\inf_{t \in A} \sup_{x \in F} x(t) - d(y) \leqslant \inf_{t \in A} y(t) \leqslant \sup_{t \in R} y(t) = 0.$$

Thus $a_2 \leqslant d(y)$.

Similarly one obtains $a_3 \leqslant d(y)$. $y \in V$ being arbitrary, we have $\text{rad}_V(F) = \text{int}_{y \in V} d(y) \geqslant c$.

Now, we show that there exists a $y_0 \in V$ with $d(y_0) \leqslant \text{rad}_V(F)$. Put

$$u(t) = \begin{cases} \sup_{x \in F} x(t) - c, & t \in S \setminus R, \\ 0, & t \in R, \end{cases}$$

$$v(t) = \begin{cases} \inf_{x \in F} x(t) + c, & t \in S \setminus R, \\ 0, & t \in R. \end{cases}$$

In a way similar to that used in the proof of Theorem 6.2 of [4] it may be shown that for all $r, s \in \mathbb{R}$ with $r < s$ we have $\{t \in S; u(t) \geqslant s\}$ non $\delta \{t \in S; v(t) < r\}$. So there is, by Theorems 2.2 and 2.3 of [4], a $y_0 \in W$ such that $u \leqslant y_0 \leqslant v$ which is equivalent to $y_0 \in V$ and $d(y_0) \leqslant c$.

Acknowledgment

I wish to thank the referee for his comments which helped to eliminate inaccuracies and errors in the original manuscript.

References

1. E. M. Alfsen and E. G. Effros, Structure in real Banach spaces, *Ann. of Math.* **96** (1972), 98–173.
2. T. Ando, Closed range theorems for convex sets and linear liftings, *Pacific J. Math.* **44** (1973), 393–410.

3. J. BLATTER, Grothendieck spaces in approximation theory, *Mem. Amer. Math. Soc.* **120** (1972).
4. J. BLATTER AND G. L. SEEVER, Interposition and lattice cones of functions, *Trans. Amer. Math. Soc.* **222** (1976), 65–96.
5. H. FAKHOURY, Projections de meilleure approximation continues dans certains espaces de Banach, *C. R. Acad. Sci. Paris Sér. A* **276** (1973), 45–48.
6. R. HOLMES, B. SCRANTON, AND J. WARD, Approximation from the space of compact operators and other *M*-ideals, *Duke Math. J.* **42** (1975), 259–269.
7. H. E. LACEY, The isometric theory of classical Banach spaces, *in* "Die Grundlagen der mathematischen Wissenschaften 208," Springer–Verlag, Berlin/Heidelberg/New York.
8. A. LIMA, Intersection properties of balls and subspaces in Banach spaces, *Trans. Amer. Math. Soc.* **227** (1977), 1–62.
9. C. OLECH, Approximation of set-valued functions by continuous functions, *Colloq. Math.* **19** (1968), 285–293.
10. A. PELCZYNSKI, A generalisation of Stone's theorem on approximation, *Bull. Acad. Polon. Sci.* **5** (1957), 105–107.
11. P. W. SMITH AND J. D. WARD, Restricted centers in subalgebras of $C(X)$, *J. Approximation Theory* **15** (1975), 54–59.
12. A. S. B. HOLLAND, B. N. SAHNEY, AND J. TZIMBALARIO, On best simultaneous approximation, *J. Approximation Theory* **17** (1976), 187–188.
13. P. D. MILMAN, On best simultaneous approximation in normed linear spaces, *J. Approximation Theory* **20** (1977), 223–238.
14. E. MICHAEL, Continuous selections, I, *Ann. of Math.* **63** (1956), 361–381.
15. Z. SEMADENI, "Banach Spaces of Continuous Functions," Warszawa, 1971.
16. J. MACH AND J. D. WARD, Approximation by compact operators on certain Banach spaces, *J. Approximation Theory* **23** (1978), 274–286.